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AN ELLIPTIC BOUNDARY VALUE PROBLEM WITH NON-DIFFERENTIABLE PARAMETERS

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§1. Introduction.

In this report a class of second order linear partial differential equations in two independent variables will be treated,

$$(1.1) \qquad \varepsilon L_1\Phi + L_2\Phi = h(x,y)$$

where L_1 is a second order uniformly elliptic p.d.o., L_2 a first order p.d.o., $h(x,y)$ a given function and ε a small positive parameter. The function Φ satisfies the differential equation in a bounded domain G in $\mathbb{R}^2$ and takes prescribed values along the boundary of G. Under certain conditions a uniform first order approximation of the solution Φ of this singular perturbation problem will be given. This kind of singular perturbations has been treated among others by Levinson [4], Vishik & Lyusternik [5], Eckhaus & de Jager [1] and Frankena [2]. In [1] an iteration procedure is given to determine an approximation of Φ up to a certain order of ε depending on the differentiability properties of the coefficients of the operator and the differentiability of the other parameters of the problem. This approximation is uniform in a convex domain G from which small neighbourhoods of the "end"-points A and B are excluded. The boundary points A and B are characterised by the fact that the characteristics of L_2 through these have no other points in common with ∂G. In their proof Eckhaus & de Jager assume that the parametric representation of the boundary is C^6 and that the other parameters are C^3 to get an approximation of order $O(\varepsilon)$.

In this report attention is paid especially to approximations of solutions of the problem for which some part of the boundary and/or the boundary value function is non-differentiable. It will appear that by means of a regularization procedure a uniform first order approximation can be given even in the case where a part of the boundary and the boundary values are required to satisfy a Hölder condition only.

Definitions:

$$L_1 = a(x,y)\frac{\partial^2}{\partial x^2} + b(x,y)\frac{\partial^2}{\partial x\,\partial y} + c(x,y)\frac{\partial^2}{\partial y^2}$$

$$+ d(x,y)\frac{\partial}{\partial x} + e(x,y)\frac{\partial}{\partial y} + f(x,y)$$

with $ac - b^2 > 0$ on $\overline{G}$ because of the ellipticity of L_1; furthermore it is assumed that $a(x,y) > 0$ on $\overline{G}$.

$$L_2 = -\frac{\partial}{\partial y} - g(x,y)$$

with $g(x,y) > 0$ on $\overline{G}$.
G is a domain in $\mathbb{R}^2$ bounded from above by

$$\{(x, \gamma_+(x)) \mid x_1 \leq x \leq x_2\}$$

and from below by

$$\{(x, \gamma_-(x)) \mid x_1 \leq x \leq x_2\} \;;$$

γ_+ and γ_- are Höldercontinuous and piecewise continuously differentiable on $[x_1,x_2]$ such that $\gamma_+(x_i) = \gamma_-(x_i)$ and $\gamma_+(x) > \gamma_-(x)$ if $x \neq x_i$ for $i = 1,2$.

The prescribed boundary values of Φ are

(1.2a) $$\Phi(x, \gamma_+(x)) = \phi_+(x)$$

(1.2b) $$(x, \gamma_-(x)) = \phi_-(x).$$

In order that the problem may certainly be uniquely solvable, it is assumed that the coefficients of L_1 and L_2, $h(x,y)$, ϕ_+ and ϕ_- are at least Höldercontinuous.

Notation:

A real function f is called $C^{m+\alpha}$ if it is m times differentiable and the m-th derivative is uniformly Höldercontinuous with exponent α; here m is a non-negative integer and α a positive real smaller than or equal to 1; sometimes the domain D of the function will be specified by writing: $f \in C^{m+\alpha}[D]$.

The Landau O-symbol will be used, but for typographical reasons we will usually write $O(\varepsilon;\lambda)$ instead of $O(\varepsilon^{\lambda})$.

The author is indebted to prof.dr. E.M. de Jager for stimulating discussions about the subject and for carefull reading of the manuscript.

§2. The maximum principle and its consequences.

In this section some lemma's will be given, which are based on the maximum principle for elliptic boundary value problems. These lemma's will be used in §§4-5 in order to prove the validity of the asymptotic approximations of the solution of the boundary value problem (1.1)-(1.2). (c.f. [1] §2).

The maximum principle may be formulated as follows:
"If a twice continuously differentiable function Φ attains a positive maximum in an interior point (x_0,y_0) of the domain G and if $f(x_0,y_0) \leq 0$ then $L_1\Phi \leq 0$ in (x_0,y_0)" (c.f. [6]).
From this well-known principle one easily derives the following lemma:

<u>Lemma 2.1</u>
If Φ and Ψ are C^2 functions on G, and if

$$f < 0 \text{ on } G$$
$$\Phi < \Psi \text{ on } \partial G$$
$$L_1\Phi > L_1\Psi \text{ in the interior of } G$$

then also

$$\Phi \leq \Psi \text{ in every point of } G.$$

proof: If $\Phi-\Psi$ attains a positive maximum in the interior of G, then $L_1[\Phi-\Psi] \leq 0$ according to the maximum principle. This is in contradiction with the assumption $L_1\Phi > L_1\Psi$ and so $\Phi-\Psi$ attains no positive maximum in the interior of G. From $\Phi-\Psi < 0$ on ∂G it follows that $\Phi-\Psi$ cannot attain positive values in G. Hence $\Phi \leq \Psi$ in G.

remark: Ψ is called a barrier function (from above) for Φ.

<u>Lemma 2.2</u> (c.f. [1] theorem III)
If Φ_ε satisfies

$$(\varepsilon L_1 + L_2)\Phi_\varepsilon = h(x,y;\varepsilon) \quad \text{for } (x,y) \in G$$

$$\Phi_\varepsilon|_{\partial G} = O(\varepsilon;\nu) \quad \text{uniformly on } \partial G$$

with $0 \leq \nu \leq 1$ and $h(x,y,\varepsilon) = O(\varepsilon;\nu)$ uniformly in G, then

$$\Phi_\varepsilon(x,y) = O(\varepsilon;\nu) \text{ uniformly in G.}$$

proof: Let y_0 be the minimum of $\gamma_-(x)$, let M be a positive number such that $|h_\varepsilon| \leq M\,\varepsilon^\nu$ on G and $|\Phi_\varepsilon|_{\partial G}| \leq M\,\varepsilon^\nu$ on ∂G and let α be a positive real number such that $1 - \varepsilon\, e(x,y) \geq \alpha$ on G if ε is small enough. Define ω by $\omega(x,y) = \{\frac{1}{\alpha}(y-y_0) + 1\}M\,\varepsilon^\nu$, then ω will be a barrier function for Φ_ε.
It is easily seen that

$$|\Phi_\varepsilon| \leq \omega \text{ on G}$$

and

$$L_\varepsilon(-\omega) \geq h_\varepsilon = L_\varepsilon\,\Phi_\varepsilon \geq L_\varepsilon\,\omega.$$

So, according to lemma (2.1) we have

$$|\Phi_\varepsilon| \leq \omega = O(\varepsilon;\nu)$$

uniformly on G.

§3. Regularization

From lemma 2.2 we see that an approximation of the solution of a boundary value problem, as defined in §1, must satisfy the differential equation and the boundary conditions at least approximately; moreover we see that the approximation, which depends functionally on the approximated parameters, must just like the solution itself be two times differentiable at least. So first a device, called regularization, will be given to approximate the boundary value functions by suitable ones.

Define the function

$$\Psi(x) = \frac{2}{3\sqrt{\pi}} \left(1 - \frac{2}{3} x^2\right)\exp - x^2,$$

then we have

(3.1) $$\int_{-\infty}^{\infty} \Psi(x)\, dx = 1$$

(3.2) $$\int_{-\infty}^{\infty} x^2\, \Psi(x)\, dx = 0$$

(3.3) $$\int_{-\infty}^{\infty} x^{2k-1}\, \Psi(x)\, dx = 0 \qquad \forall k \in \mathbb{N}$$

The regularization of a uniformly bounded continuous function f is now defined by

(3.4) $$f_\varepsilon(p) = \varepsilon^{-\lambda} \int_{-\infty}^{\infty} f(x)\, \Psi\left(\frac{p-x}{\varepsilon^\lambda}\right) dx$$

where λ is a positive parameter; this parameter will be chosen later on to optimalize estimates of the difference between solutions and approximations. The regularized function f_ε is a C^∞-function that converges uniformly to f if ε tends to zero (c.f. [3] theorem 1.2.1) and it is easily seen by partial integration that

(3.5) $$(f_\varepsilon)' = (f')_\varepsilon$$

if f is differentiable.

Furthermore we have the following uniform estimates of f_ε and its derivatives if $f \in C^{k+\alpha}[\mathbb{R}]$:

$$(3.6)\qquad |f^{(i)} - f_\varepsilon^{(i)}| = \begin{cases} O(\varepsilon;\lambda(k+\alpha-i)) & \text{if } 0 \leq k-i \leq 3 \\ O(\varepsilon;4\lambda) & \text{if } k-i \geq 4 \end{cases}$$

$$(3.7)\qquad |f_\varepsilon^{(i)}| = O(\varepsilon;\ \min\{0,k-i+\alpha\})$$

remark: It is possible to choose instead of $\Psi(x)$ a product of another polynomial and $\exp{-x^2}$, such that integrals containing higher even powers of x also vanish and such that the estimates analogous to (3.6) become better if $k-i \geq 4$. For the present purpose however they are not needed for we do not regularize boundary value functions that are more than three times differentiable; singular perturbation problems with boundary value functions of class C^3 can be treated without regularization, as is done for instance in the approach of Eckhaus & de Jager.

The proof of formula (3.6) will be given for $i = 0$ and $k = 1$ and $k = 4$; formula (3.7) will be proved for $k = 0$ and $i = 1$. If $k-i+\alpha \geq 0$, (3.7) is obvious and the proofs of the other cases of (3.6) and (3.7) are analogous to the proofs given.

proof of 3.6

If $f \in C^{1+\alpha}$, then

$$f(p+h) = f(p) + h\, f'(p+\theta h) \qquad (0 < \theta < 1)$$

and so

$$|f(p+h) - f(p) - h\, f'(p)| \leq |h|\ |f'(p+\theta h) - f'(p)| = O(h;1+\alpha).$$

Hence

$$f_\varepsilon(p) - f(p) = \varepsilon^{-\lambda} \int_{-\infty}^{\infty} (f(x) - f(p)\ \Psi\left(\frac{x-p}{\varepsilon^\lambda}\right) dx$$

$$= \varepsilon^{-\lambda} \int_{-\infty}^{\infty} (f(t+p) - f(p)\ \Psi(\varepsilon^{-\lambda}t)dt.$$

Because of (3.3)

(3.8) $$f_\varepsilon(p) - f(p) = \varepsilon^{-\lambda} \int_{-\infty}^{\infty} (f(t+p) - f(p) - t\, f'(p))\, \Psi(\varepsilon^{-\lambda}t)dt.$$

For a certain positive constant K we then have

$$|f_\varepsilon(p) - f(p)| \leq K\, \varepsilon^{-\lambda} \int_{-\infty}^{\infty} |t|^{1+\alpha} |\Psi(\varepsilon^{-\lambda}t)|dt$$

$$\leq K\, \varepsilon^{\lambda(1+\alpha)} \int_{-\infty}^{\infty} |s|^{1+\alpha}\, |\Psi(s)|\, ds$$

$$= O(\varepsilon;\lambda(1+\alpha)).$$

Because of the fact that

$$\int_{-\infty}^{\infty} t^4\, f^{(4)}(p)\, \Psi(\varepsilon^{-\lambda}t)dt \neq 0,$$

the analogue of (3.8) is not true and we get

$$|f_\varepsilon(p) - f(p)| \leq K\, \varepsilon^{-\lambda} \int_{-\infty}^{\infty} |t^4|\, |\Psi(\varepsilon^{-\lambda}t)|dt = O(\varepsilon;4\lambda).$$

proof of (3.7)

By substitution of $f \equiv 1$ in 3.5 we get

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial p}\, \Psi(p-x)dx = 0.$$

Hence, if $f \in C^\alpha[\mathbb{R}]$,

$$f'_\varepsilon(p) = \varepsilon^{-\lambda} \int_{-\infty}^{\infty} f(x)\, \frac{\partial}{\partial p}\, \Psi\left(\frac{p-x}{\varepsilon^\lambda}\right)\, dx$$

$$= \varepsilon^{-\lambda} \int_{-\infty}^{\infty} (f(p+t) - f(p))\, \frac{d}{dt}\, \Psi(\varepsilon^{-\lambda}t)\, dt$$

and there exists a positive constant K, such that

$$|f'_\varepsilon(p)| \leq K\, \varepsilon^{-\lambda} \int_{-\infty}^{\infty} |t^\alpha\, \varepsilon^{-\lambda}\, \Psi'(\varepsilon^{-\lambda}t)|\, dt = O(\varepsilon;\lambda(\alpha-1)).$$

We see that the derivatives of a regularized function need not be bounded if ε tends to zero; in the following sections however we will get an additional multiplicative factor of order $O(\varepsilon)$, which will give compensation, just when we need it.

§4. Asymptotic solutions for non-differentiable boundary values.

Let us return to the problem defined in (1.1-2) and assume that the coefficients of the operator and h are $C^3(\overline{G})$

(4.1a) $\gamma_- \in C^3[x_1,x_2]$

(4.1b) $\gamma_+ \in C^4[x_1,x_2]$

(4.2) ϕ_+ and $\phi_- \in C^{k+\alpha}[x_1,x_2]$

with $k = 0,1,2$ and $0 < \alpha \leq 1$;

it is noted that the right and the left derivative of γ_+ and γ_- in x_1 and x_2 respectively must be finite.

In §3 of their paper ([1]) Eckhaus & de Jager give an asymptotic approximation of the solution Φ of the problem assuming much stronger differentiability conditions. In this report their proof will be simplified by using another local coordinate system in the boundary layer. Moreover with the aid of approximations of the boundary value functions by regularization a method for approximating Φ will be given in the case that ϕ_+ and ϕ_- are non-differentiable. Later on also the differentiability conditions of the lower boundary will be weakened.

Without loss of generality we may assume from now on that the upper boundary is a straight line; otherwise a coordinate transformation may be performed, namely

$$(x,y) \longmapsto (u,v) = (x, y - \gamma_+(x)).$$

The functional determinant of this transformation equals 1, so the elliptic character of the operator L_1 remains unchanged; all other conditions also remain unchanged, except for the fact that the coefficient of $\frac{\partial}{\partial y}$ in L_1 is C^2 and no longer C^3, but this will not affect the proof.
It will appear that a boundary layer must be constructed along the upper boundary. With this assumption about the upper boundary the stretching of the coordinates in the boundary layer has become very simple, namely $y = \varepsilon\eta$.

In the boundary layer the operator $\varepsilon L_1(x,\varepsilon\eta) + L_2(x,\varepsilon\eta)$, in which the stretched coordinate is substituted, can be expanded into powers of ε.

The stretching $y = \varepsilon\eta$ gives $\frac{\partial}{\partial y} = \frac{1}{\varepsilon}\frac{\partial}{\partial\eta}$
and expansion of coefficients of L_1 and L_2 then yields

$$\begin{aligned}
b(x,\varepsilon\eta) &= b_0(x) + \varepsilon\eta\, b_1(x,\varepsilon\eta)\\
c(x,\varepsilon\eta) &= c_0(x) + \varepsilon\eta\, c_1(x) + \varepsilon^2\eta^2\, c_2(x,\varepsilon\eta)\\
e(x,\varepsilon\eta) &= e_0(x) + \varepsilon\eta\, e_1(x,\varepsilon\eta)\\
g(x,\varepsilon\eta) &= g_0(x) + \varepsilon\eta\, g_1(x,\varepsilon\eta)\ .
\end{aligned}$$

Here b_0, c_0, e_0 and g_0 are $C^3[x_1,x_2]$,

c_1 is $C^2[x_1 x_2]$

b_1, e_1 and g_1 are $C^2[\overline{G}]$,

c_2 is $C'[\overline{G}]$.

The operator $\varepsilon L_1 + L_2$, expressed in local coordinates, can be expanded into powers of ε:

$$\varepsilon L_1 + L_2 = \frac{1}{\varepsilon} M_0 + M_1 + \varepsilon M_2$$

where

$$M_0(x,\eta) = c_0(x)\frac{\partial}{\partial\eta^2} - \frac{\partial}{\partial\eta}$$

$$M_1(x,\eta) = \eta\ c_1(x)\frac{\partial^2}{\partial\eta^2} + 2\ b_0(x)\frac{\partial}{\partial x\partial\eta} + e_0(x)\frac{\partial}{\partial\eta} - g_0(x)$$

$$M_2(x,\eta) = a(x,\varepsilon\eta)\frac{\partial^2}{\partial x^2} + 2\eta\ b_1(x,\varepsilon\eta)\frac{\partial^2}{\partial x\partial\eta} + \eta^2 c_2(x,\varepsilon\eta)\frac{\partial^2}{\partial\eta^2}$$
$$+ d(x,\varepsilon\eta)\frac{\partial}{\partial x} + \eta\ e_1(x,\varepsilon\eta)\frac{\partial}{\partial\eta} + f(x,\varepsilon\eta) - \eta\ g_1(x,\varepsilon\eta).$$

Because of the uniform ellipticity of L_1 and the positivity of $a(x,y)$ in $\overline{G}$, we also have $c(x,y) > 0$ and so $c_0(x) > 0$ on $[x_1,x_2]$.

Theorem I.

Under the conditions (4.1) and (4.2) and, moreover, if $\gamma_-'(+x_1) \neq 0$ and $\gamma_-'(-x_2) \neq 0$, a function Φ_0 can be constructed such, that for the solution Φ of (1,1-2) we have

$$\Phi - \Phi_0 = O(\varepsilon;\ \min\{1, \frac{k+\alpha}{2}\})$$

proof:
The proof is based on the construction of a function Φ_0 that approximately fulfils the boundary conditions and for which $(\varepsilon\ L_1+L_2)\ (\Phi-\Phi_0)$ is uniformly small; with lemma 2.2 it then follows that $\Phi-\Phi_0$ is small everywhere in G.

Extend ϕ_+ and ϕ_- to be functions on $\mathbb{R}$ by

$$\phi_{\pm}(x) \equiv \phi_{\pm}(x_1) \text{ if } x < x_1$$
$$\phi_{\pm}(x) \equiv \phi_{\pm}(x_2) \text{ if } x > x_2$$

and define $\phi_{\varepsilon+}$ and $\phi_{\varepsilon-}$ as their regularizations.
Next we define w(x,y) to be the solution of the reduced equation of (1.1)

$$-\ (\frac{\partial}{\partial y} + g(x,y))w(x,y) = h(x,y)$$

which equals $\phi_{\varepsilon-}$ at the lower boundary of G.
If we define p and q by

$$p(x,y,t) = \int_t^y h(x,\eta)\ \exp\{\ -\int_\eta^y\ y(x,\zeta)d\zeta\}\ d\eta\ ,$$

we have

(4.3) $\quad w(x,y) = \phi_{\varepsilon-}(x) - p(x,y,\ \gamma_-(x)) - \phi_{\varepsilon-}\ q(x,y,\gamma_-(x)).$

From this we see that w is linearly dependent on $\phi_{\varepsilon-}$, so there exists a positive constant K for which

(4.4) $\quad |\varepsilon\ L_1 w| \leq \varepsilon\ K\ (|\phi_{\varepsilon-}| + |\phi_{\varepsilon-}'| + |\phi_{\varepsilon-}''|)\ ,$

and because of (3.7)

(4.5) $\quad \varepsilon\ L_1 w\ = O(\varepsilon;\ 1 + \min\{0,\ \lambda(k+\alpha-2\})$

applies uniformly on $\overline{G}$.

In general the function $w(x,y)$ does not fulfil the upper boundary condition and an adaptation should be made in a neighbourhood of the upper boundary.
If we define the function $\rho(x)$ by

$$\rho(x) = \phi_{\varepsilon+}(x) - w(x,0),$$

we have obtained a new boundary value problem, which must be fulfilled by the correction function, namely

$$\varepsilon L_1 v(x,y) + l_2 v(x,y) = 0$$

with boundary conditions

$$v(x,0) = \rho(x) \qquad \text{(at the upper boundary)}$$
$$v(x, \gamma_-(x)) = 0 \qquad \text{(at the lower boundary)-}$$

To approximate the solution we stretch the ordinate $y = \varepsilon\eta$, we develop $v(x,\varepsilon\eta) = v_0(x,\eta) + \varepsilon\, v_1(x, \eta) + \ldots$,
and solve the zeroth and first order reduced equations,

$$M_0 v_0 = 0$$
$$\text{with } v_0(x,0) = \rho(x) \text{ and } \lim_{\eta\to-\infty} v_0(x,\eta) = 0$$

and

$$M_0 v_1 = - M_1 v_0$$
$$\text{with } v_1(x,0) = 0 \text{ and } \lim_{\eta\to-\infty} v_1(x,\eta) = 0 \ .$$

The solutions are

$$v_0(x,\eta) = \rho(x) \exp \frac{\eta}{c_0(x)} \tag{4.6}$$

$$v_1(x,\eta) = \{(b_0 c_0' - \tfrac{1}{2} c_0 c_1)\rho\, c_0^{-3}\eta + (g_0 c_0 + c_1 - e_0)\rho\, c_0^{-1} - 2\, b_0 c_0^{-1} \rho'\}\eta \exp \frac{\eta}{c_0} \ . \tag{4.7}$$

It is obvious that ρ is linearly dependent on $\phi_{\varepsilon+}$ and $\phi_{\varepsilon-}$; hence v_0 is linearly dependent on $\phi_{\varepsilon+}$ and $\phi_{\varepsilon-}$ and v_1 on $\phi_{\varepsilon\pm}$ and $\phi'_{\varepsilon\pm}$.

In the same way as in (4.4) and (4.5) we may conclude

(4.8a) $$|M_2\ v_0| \leq K\ (|\phi_{\varepsilon+}| + |\phi_{\varepsilon-}| + |\phi'_{\varepsilon+}| + |\phi'_{\varepsilon-}| + |\phi''_{\varepsilon+}| + |\phi''_{\varepsilon-}|)$$

$$= O(\varepsilon;\ \min\{0,\lambda(k+\alpha-2\})$$

(4.8b) $$|M_1\ v_1| \leq K\ (|\phi_{\varepsilon+}| + \ldots\ldots + \phi''_{\varepsilon-}) = O(\varepsilon;\ \min\{0,\lambda(k+\alpha-2\})$$

(4.8c) $$|M_2\ v_1| \leq K\ (|\phi_{\varepsilon+}| + \ldots\ldots + |\phi'''_{\varepsilon}|) = O(\varepsilon;\ \min\{0,\lambda(k+\alpha-3)\}).$$

If we define Φ_0 as

$$\Phi_0(x,y) = w(x,y) + v_0(x, \frac{y}{\varepsilon}) + \varepsilon\ v_1(x, \frac{y}{\varepsilon})$$

then it is clear that

(4.9) $$(\varepsilon\ L_1+L_2)\left[\Phi-\Phi_0\right] = \varepsilon\ L_1 w + \varepsilon\ M_2\ v_0 + \varepsilon\ M_1\ v_1 + \varepsilon^2\ M_2\ v_1$$

$$= O(\varepsilon;\ \min\{1,\lambda(k+\alpha-2) + 1,\ \lambda(k+\alpha-3) + 2\}).$$

This estimate, necessary for the application of lemma 2.2, is valid within the domain G.

Next we give an estimate for $\Phi - \Phi_0$ at the boundary ∂G.

At the upper boundary we have

$$\Phi_0(x,0) = w(x,0) + v_0(x,0) + \varepsilon\ v_1(x,0) = \phi_{\varepsilon+}(x)$$

and from (3.6) it follows that

(4.10) $$\Phi(x,0) - \Phi_0(x,0) = \phi_+(x) - \phi_{\varepsilon+}(x) = O(\varepsilon;\lambda(k+\alpha)).$$

At the lower boundary we have the relation

$$\Phi_0(x,\gamma_-(x)) = w(x,\gamma_-(x)) + v_0(x,\tfrac{1}{\varepsilon}\gamma_-(x)) + \varepsilon\, v_1(x,\tfrac{1}{\varepsilon}\gamma_-(x))$$

Here

$$w(x, \tfrac{1}{\varepsilon}\,\gamma_-(x)) = \phi_{\varepsilon-}(x) = \phi_-(x) + O(\varepsilon;\lambda(k+\alpha).$$

If $\alpha > 0$ we have the estimate

$$x^{\alpha} \exp \frac{x}{\varepsilon} = O(\varepsilon;\alpha) \text{ uniformly for } x \leq 0, \tag{4.11}$$

so the factor $\dfrac{\gamma_-(x)}{\varepsilon} \exp \dfrac{\gamma_-(x)}{\varepsilon\; c_0(x)}$, contained in v_1, is uniformly bounded, for $\gamma_-(x) \leq 0$ and $c_0(x)$ is strictly positive. Furthermore, as was already mentioned, v_1 is linearly dependent on $\phi_{\varepsilon\pm}$ and $\phi'_{\varepsilon\pm}$; hence we have the relation

$$|v_1| \leq K(|\phi_{\varepsilon+}| + |\phi_{\varepsilon-}| + |\phi'_{\varepsilon+}| + |\phi'_{\varepsilon-}|) =$$

$$= O(\varepsilon;\ \min\{\lambda(k+\alpha-1),\ 0\}).$$

Finally we need an estimate of

$$v_0(x,\tfrac{1}{\varepsilon}\,\gamma_-(x)) = \{\phi_{\varepsilon+}(x) - \phi_{\varepsilon-}(x) + p(x,0,\gamma_-(x)) + \\ + \phi_{\varepsilon-}\, q(x,0,\gamma_-(x))\} \exp \frac{\gamma_-(x)}{\varepsilon\; c_0(x)}\ . \tag{4.12}$$

From the definition of p and q it is easily seen that there exists a constant K_1, such that

$$|p(x,0,\gamma_-(x)) + \phi_{\varepsilon-} q(x,0,\gamma_-(x))| \leq K_1\ |\gamma_-(x)|$$

and with (4.11) it then follows that

$$(p + \phi_{\varepsilon-} q) \exp \frac{\gamma_-(x)}{\varepsilon\; c_0(x)} = O(\varepsilon).$$

From assumption (4.2) and formula (3.6) it follows that

$$\phi_{\varepsilon+}(x) - \phi_{\varepsilon-}(x) = (\phi_{\varepsilon+}(x) - \phi_+(x)) + (\phi_+(x) - \phi_+(x_i)) +$$

$$+ (\phi_-(x_i) - \phi_-(x)) + (\phi_-(x) - \phi_{\varepsilon-}(x))$$

$$= O(\varepsilon;\lambda(k+\alpha)) + O(x-x_i;\ \min\{1,k+\alpha\})$$

for $i = 1$ and $i = 2$; because it was assumed that $\gamma'(x_i) \neq 0$ there exists a constant c, such that

$$-\gamma_-(x_i) \geq c|x-x_i| \qquad (i = 1,2)$$

and with (4.11) it then follows that

$$(\phi_{\varepsilon+}(x) - \phi_{\varepsilon-}(x) \exp \frac{\gamma_-(x)}{\varepsilon\ c_0(x)} = O(\varepsilon;\min\{1,k+\alpha,\ \lambda(k+\alpha)\}).$$

Combining these, we may conclude that

$$v_0(x,\tfrac{1}{\varepsilon}\gamma_-(x)) = O(\varepsilon;\ \min(1,k+\alpha,\ \lambda(k+\alpha)).$$

Hence we have

(4.13) $\Phi_0(x,\gamma_-(x)) = \phi_-(x) + O(\varepsilon;\min\{1,k+\alpha,\ \lambda(k+\alpha),\ \lambda(k+\alpha-1) + 1\}).$

By application of lemma 2.2 to the formulae (4.9), (4.10) and (4.13) we obtain

(4.14) $\Phi - \Phi_0 = O(\varepsilon;\min\{1,k+\alpha,\ \lambda(k+\alpha),\ \lambda(k+\alpha-2) + 1,\ \lambda(k+\alpha-3) + 2\}).$

The maximum of the exponent is 1 if $k = 2$ and $\frac{1}{1+\alpha} < \lambda < \frac{1}{1-\alpha}$ and is $\frac{k+\alpha}{2}$ if $k = 0$ or $k = 1$ and $\lambda = \frac{1}{2}$, so finally we have

$$\Phi - \Phi_o = O(\varepsilon;\min\{1,\ \frac{k+\alpha}{2}\})$$

uniformly on $\overline{G}$.

Remarks:

1) For the proof of the estimate (4.9) it is necessary that Φ_0 contains the term εv_1, but since

$$\varepsilon v_1 = O(\varepsilon;\min\{\lambda(k+\alpha-1) + 1,1\})$$

$w(x,y) + v_0(x,\frac{y}{\varepsilon})$ is as good an approximation of Φ as Φ_0.

2) In the proof of theorem I we had to use regularized boundary value functions, but from (3.6) it is easily seen that if $\tilde{w}$ and $\tilde{v}_0$ are defined by

$$\tilde{w}(x,y) = \phi_-(x) - p(x,y,\gamma_-(x)) - \phi_-(x)\; q(x,y,\gamma_-(x))$$

and

$$\tilde{v}_0(x,\eta) = \{\phi_+(x) - w(x,0)\} \quad \exp \frac{\eta}{x_0(x)} \quad ,$$

then $\tilde{w}(x,y) + \tilde{v}_0(x,\frac{y}{\varepsilon})$ is as good an approximation of Φ as Φ_0 and $w(x,y) + v_0(x,\frac{y}{\varepsilon})$.

§5. Non-differentiable lower boundary.

The method of proving asymptotic properties with use of regularization can also be applied, if the lower boundary is non-differentiable. So we arrive at the following theorem:

Theorem II.

Under the same conditions as theorem I, except that we now assume for γ_-,

(5.1)
a. $\gamma_- \in C^{l+\beta}|x_1,x_2|$ $(l = 0,1,2,3$ and $0 < \beta \leq 1)$
b. γ_- is at least piecewise continuously differentiable.
c. there exists a positive constant c, such that $\gamma_-(x) < -c|x - x_i|$ in a neighbourhood of the points $(x_1,0)$ and $(x_2,0)$,

a function Φ_0 can be constructed, such that we have for the solution of (1.1-2).

$$\Phi - \Phi_0 = O(\varepsilon; \min\{1, \frac{k+\alpha}{2}, \frac{l+\beta}{2}\}).$$

proof:

The proof goes along the same lines as the previous one and therefore it will not be repeated completely. Differences arise in the estimates (4.4), (4.8) and (4.13); only these will be reconsidered.

Apart from the boundary values $\phi_{\pm}$ the lower boundary γ_- has to be regularized also now, which results in $\phi_{\varepsilon\pm}$ and $\gamma_{\varepsilon-}$. The functions w, v_0 and v_1 are obtained by the same calculations as in §4, only γ_- has to be replaced by $\gamma_{\varepsilon-}$. This results in

$$w(x,y) = \phi_{\varepsilon-}(x) - p(x,y,\gamma_{\varepsilon-}(x)) - \phi_{\varepsilon-}(x)\, q(x,y,\gamma_{\varepsilon-}(x))$$

$$v_0(x,\eta) = (\phi_{\varepsilon+}(x) - w(x,0)) \exp \frac{\eta}{c_0(x)} = \rho(x) \exp \frac{\eta}{c_0(x)}$$

$$v_1(x,\eta) = \{(b_0c_0' - \tfrac{1}{2} c_0c_1)\rho\, c_0^3\eta + (g_0c_0 + c_1 - e_0)\rho\, c_0^{-1} - 2\, b_0c_0^{-1}\, \rho'\}\eta \exp \frac{\eta}{c_0}$$

and again

$$\Phi_0(x,y) = w(x,y) + v_0(x,\tfrac{y}{\varepsilon}) + \varepsilon\, v_1(x,\tfrac{y}{\varepsilon}) .$$

It is easily seen that $p(x,y,t)$ and $q(x,y,t)$ are C^3 functions with uniformly bounded third derivatives, hence by differentiation it follows that

(5.2a) $\frac{\partial w}{\partial x} = O(|\phi'_{\varepsilon-}| + |\gamma'_{\varepsilon-}|)) = O(\varepsilon;\ \lambda \min\{0, k+\alpha-1, l+\beta-1\})$

(5.2b) $$\frac{\partial^2 w}{\partial x^2} = O(|\phi'_{\varepsilon-}| + |\gamma'_{\varepsilon-}| + |\phi''_{\varepsilon-}| + |\gamma''_{\varepsilon-}| + |\gamma'_{\varepsilon-}|^2 + |\phi'_{\varepsilon}\gamma'_{\varepsilon}|)$$

$$= O(\varepsilon;\lambda \min\{0,\ k+\alpha-2,\ l+\beta-2\})$$

(5.2c) $$\frac{\partial^3 w}{\partial x^2} = O(|\phi'_{\varepsilon-}| + \ldots\ldots + |\gamma'''_{\varepsilon-}| + |\phi'_{\varepsilon-}\gamma'_{\varepsilon-}| + \ldots\ldots + |\gamma'_{\varepsilon-}|^3) =$$

$$= O(\varepsilon;\lambda \min\{0,\ k+\alpha-3,\ l+\beta-3\}).$$

Furthermore v_0 is linearly dependent on $\phi_{\varepsilon+}$ and $w(x,0)$ and v_1 on $\phi_{\varepsilon+}$, $\phi'_{\varepsilon+}$, $w(x,0)$ and $\frac{\partial}{\partial x} w(x,0)$; hence the analogue of (4.9) is

(5.3) $$(\varepsilon\ L_1+L_2)\ (\Phi-\Phi_0) = O(\varepsilon;\ \min\{1, \lambda(k+\alpha-2) + 1,\ \lambda(l+\beta-2) + 1,$$
$$\lambda(k+\alpha-3) + 2,\ \lambda(l+\beta-3) + 2\}) .$$

At the upper boundary we again have

(5.4) $$\Phi_0(x,0) = w(x,0) + v_0(x,0) + \varepsilon\ v_1(x,0) = \phi_{\varepsilon+}(x)$$
$$= \phi_+(x) + O(\varepsilon;\ \lambda(k+\alpha))$$

At the lower boundary we have

$$\Phi_0(x,0) = w(x,\gamma_-(x)) + v_0(x,\tfrac{1}{\varepsilon}\ \gamma_-(x)) + \varepsilon\ v_1(x,\tfrac{1}{\varepsilon}\ \gamma_-(x)).$$

Because of (4.11), (5.2a) and the fact that v_1 depends linearly on $\phi_{\varepsilon-}$, $\phi'_{\varepsilon-}$, w and w', we have the relation

$$v_1 = O(\varepsilon;\ \min\{\lambda(k+\alpha-1),\ \lambda(1+\beta-1),\ 0\}) \quad .$$

From the uniform boundedness of $\frac{\partial w}{\partial y}$ and from (3.6) it follows that

$$\begin{aligned} w(x,\gamma_-(x)) &= w(x,\gamma_{\varepsilon-}(x)) + O(\gamma_-(x) - \gamma_{\varepsilon-}(x)) \\ &= \phi_{\varepsilon-}(x) + O(\varepsilon;\lambda(1+\beta)) \\ &= \phi_-(x) + O(\varepsilon;\lambda\ \min\{1+\beta,\ k+\alpha\}). \end{aligned}$$

Finally we need an estimate of

$$\begin{aligned} v_0(x,\tfrac{1}{\varepsilon}\gamma_-(x)) &= \{\phi_{\varepsilon+}(x)-\phi_{\varepsilon-}(x) + p(x,0,\gamma_{\varepsilon-}(x))+\phi_{\varepsilon-}\ q(x,0,\gamma_{\varepsilon-}(x))\}\ \exp\frac{\gamma_-(x)}{\varepsilon\ c_0(x)} \\ &= \{\phi_{\varepsilon+}(x)-\phi_{\varepsilon-}(x) + p(x,0,\gamma_-(x))+\phi_{\varepsilon-}\ q(x,0,\gamma_-(x))\}\ \exp\frac{\gamma_-(x)}{\varepsilon\ c_0(x)} \\ &\quad + O(\gamma_{\varepsilon-}(x) - \gamma_-(x)). \end{aligned}$$

We see that the estimate of this is exactly the same as the estimate of (4.12) except for the term $O(\gamma_\varepsilon - \gamma_-)$.
Hence we may conclude that

$$v_0(x,\tfrac{1}{\varepsilon}\gamma_-(x)) = O(\varepsilon;\ \min\{1,k+\alpha,\ \lambda(k+\alpha),\ \lambda(1+\beta)\}).$$

Combining these we have

$$(5.5) \qquad \Phi_0(x,\gamma_-(x)) = O(\varepsilon;\ \min\{1,k+\alpha,\ \lambda(k+\alpha),\ \lambda(1+\beta),\ \lambda(k+\alpha-1) + 1,\ \lambda(1+\beta-1) + 1\}).$$

Application of lemma 2.2 to (5.3), (5.4) and (5.5) and the optimal choice of λ, namely $\lambda = \frac{1}{2}$, yields the result

$$\Phi - \Phi_0 = O(\varepsilon;\ \min\{1,\ \tfrac{k+\alpha}{2},\ \tfrac{1+\beta}{2}\}).$$

example: Let G be bounded by $y \equiv 0$ and $y = -(1-|x|)^{1/p}$, with $p \geq 1$, and let the boundary conditions be

$$\phi_+(x) = 0 \text{ and } \phi_-(x) = x^2 \sin \frac{\pi}{x}$$

then Φ_0 is a uniform approximation of the solution of the boundary value problem with an error of order $O(\varepsilon; \frac{1}{2p})$.

§6. Conclusions.

In the preceding sections we have seen an application of mollifiers to asymptotic theory. If the upper boundary (i.e. the boundary of G before it is transformed to a straight line) is non-differentiable, there is no point in regularizing it with this method. The original boundary must remain within the boundary layer along its regularization, for otherwise the difference between the approximation and the upper boundary condition would be too large. Hence, if γ_+ is C^α with $0 < \alpha \leq 1$, $\gamma_+ - \gamma_{\varepsilon+}$ must be of an order smaller than $O(\varepsilon)$ and so $\lambda\alpha$ must be greater than 1. But then it is impossible to get a positive order of ε in the estimate of e.g. $\varepsilon L_1 w$. The coefficient of the term $\frac{\partial}{\partial y}$ of the operator L_1, after application of transformation $(x,y) \longmapsto (x, y-\gamma_{\varepsilon+}(x))$, linearly contains the second derivative of $\gamma_{\varepsilon+}$; so, if the upper boundary was regularized before its transformation, this coefficient is of order $O(\varepsilon; \lambda(\alpha-2))$. Hence $\varepsilon L_1 w$ is at least of order $O(\varepsilon; \lambda(\alpha-2) + 1)$ and it is obvious that we cannot have both $\lambda\alpha > 1$ and $\lambda(\alpha-2) + 1 > 0$.

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